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Linear codes arising from the Gale transform of distinguished subsets of some projective spaces

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ABSTRACT

Applying the Gale transform on certain linear and non-linear geometrical objects, and studying the orbits under the action of the associated automorphism groups in the higher-dimensional space, we construct some families of cap codes and other structures admitting the same automorphism groups.

1. Introduction

The Gale transform of a set $\mathcal{T}$ consisting of γ labelled points of a projective space $\mathrm{PG}(r, q)$ is an involution, defined up to automorphisms, which maps $\mathcal{T}$ into a set $\mathcal{T}'$ consisting of γ labelled points of $\mathrm{PG}(s, q)$ with $\gamma = r + s + 2$.

The simplest way to define the Gale transform of a set of points is in terms of projective coordinates. Choose homogeneous coordinates in such a way that the coordinates of the points of $\mathcal{T}$ are the rows of the matrix

$$\begin{pmatrix} I_{r+1} \\ A \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix and A is an $(s + 1) \times (r + 1)$ matrix. Then, the Gale transform of $\mathcal{T}$ is the set $\mathcal{T}'$ consisting of the points of $\mathrm{PG}(s, q)$ whose homogeneous coordinates are the rows of the matrix

$$\begin{pmatrix} {}^{T}\!A \\ I_{s+1} \end{pmatrix},$$

where ${}^{T}\!A$ is the transpose matrix of A.

The Gale transform has many geometrical and group-theoretical properties which make it a valuable tool in several disciplines such as optimization, coding theory and algebraic geometry. Here we list some recent results that will be useful. The interested reader is referred to [3,4] for proofs of the results and further details on the Gale transform.

Lemma 1.1. *If ℓ is a line in some projective space $\mathrm{PG}(r, q)$ and $\mathcal{T} \subseteq \ell$, with $|\mathcal{T}| = r + s + 2$, then the Gale transform $\mathcal{T}'$ of $\mathcal{T}$ is contained in the unique normal rational curve of $\mathrm{PG}(s, q)$ containing the fundamental frame.*

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Lemma 1.2. *The Gale transform of a k-cap in a projective space* PG(r, q), $k \geq r + 4$, *is a k-cap in* PG$(k - r - 2, q)$.

Lemmas 1.1 and 1.2 together yield the following crucial result.

Theorem 1.3. *Let* $\mathcal{T}$ *be any set consisting of k points in* PG(r, q), $r \geq 2$ *and* $k \geq r + 4$. *Then the Gale transform* $\mathcal{T}'$ *of* $\mathcal{T}$ *is a k-cap in* PG$(k - r - 2, q)$.

When constructing an error correcting code, it is convenient to have some control on its automorphism group. The following result generalizing [3, Proposition 2.5] provides a useful tool for this task.

Theorem 1.4. *Let* $\mathcal{T}$ *be any subset of* PG(r, q) *consisting of at least* $k = r + 4$ *points and* $\mathcal{T}'$ *its Gale transform. Then* $\mathcal{T}$ *and* $\mathcal{T}'$ *have isomorphic collineation groups.*

Proof. Let $\mathcal{T} = \{p_1, \ldots, p_k\}$. The $(r + 1) \times k$ matrix $(p_1 \ldots p_k)$ determines a vector subspace $V = V(r + 1, q)$ in the vector space $V(k, q)$. This is due to the fact that $\mathcal{T}$ always contains the fundamental frame of PG(r, q). The Gale transform of $\mathcal{T}$ produces a new set in PG$(k - r - 2, q)$ containing the fundamental frame which generates the orthogonal complement $V^{\perp}$ of V in $V(k, q)$. The theorem now follows from [11, Theorem 2.1]. □

The main idea in this paper is to use the Gale transform of some remarkable objects in projective spaces to construct other interesting objects and possibly study the associated linear codes.

An interesting question that can be posed is the following. When does it happen that the Gale transform of a subset of the projective space is embedded in a Veronese variety? This fact is relevant because a Veronese variety is always a cap and its extensions could give rise to interesting cap codes. In this paper we have studied the cases of a classical unital in a projective plane of order $q = 4$, a distinguished 1-set in PG(3, 4), a Baer subplane in PG(2, 4) and a Baer subplane in PG(2, 9).

2. The Gale transform of $\mathcal{H}(2, 4)$ and its code

A classical unital $\mathcal{H}(2, q)$ in a projective plane PG$(2, q)$ of square order q is the set of all absolute points of a non-degenerate Hermitian form. It consists of $q\sqrt{q} + 1$ points such that each line meets $\mathcal{H}(2, q)$ at either 1 (tangent) or $\sqrt{q} + 1$ points (secant). The collineation group preserving $\mathcal{H}(2, q)$ in PGL$(3, q)$ is the subgroup PGU$(3, q)$, see [8] and also [9, Theorem 2.50] or [6, Example A.9].

From now on we shall focus on the projective plane PG(2, 4) over the field GF(4) $= \{0, 1, \omega, \omega^2\}$, with $\omega^2 + \omega + 1 = 0$.

The projective plane PG(2, 4) contains exactly 280 unitals, 18 of which pass through the fundamental points (1, 0, 0), (0, 1, 0) and (0, 0, 1). Each of these $\mathcal{H}(2, 4)$'s admits one tangent and four secants through each of its points. Assume that $\mathcal{H} = \mathcal{H}(2, 4)$ has equation

$$\omega(XY^2 + XZ^2 + YZ^2) + \omega^2(X^2Y + X^2Z + Y^2Z) = 0.$$

A quick computation [2] provides the point set of $\mathcal{H}$, as in Table 5. Its Gale transform is the set $\mathcal{H}'$ – whose points constitute a 9-cap in PG(5, 4), see Table 6 – admitting PGU(3, 4) as automorphism group.

Actually, the group $G =$ PGU(3, 4) acting on $\mathcal{H}'$ has, among several others, the following orbits on points of PG(5, 4):

- the point set $\mathcal{H}'$ itself;
- two orbits, say $\mathcal{R}$ and $\mathcal{T}$, of size 12;
- an orbit $\mathcal{S}$ of size 9.

It turns out that one of the orbits of size 12, say $\mathcal{R}$, is a cap in PG(5, 4) that together with $\mathcal{H}'$ gives rise to a 21-cap $\mathcal{V}$ (see Tables 6 and 7). The other orbit $\mathcal{T}$ of size 12, together with the other orbit $\mathcal{S}$ of size 9, gives rise to a plane, say π. It turns out that G is reducible in its action on the points of PG(5, 4) (see Tables 8 and 9). Looking at the list of maximal subgroups of PSL$(6, q)$ as presented by Kleidman [10], we conclude that the stabilizer of the 21-cap $\mathcal{V}$ is the symmetric square of PSL(3, 4), see
[1, Proposition 4]; hence $\mathcal{V}$ is a Veronese surface with nucleus π, see [7, Chapter 25]. In Table 10 we have reported the 21 conics embedded in $\mathcal{V}$.

The plane π admits a regular hyperoval $\mathcal{C}$ consisting of six points and preserved by the group $A_6 \cong$ PSL(2, 9), see [12]. For instance, such a hyperoval in π is given by

$$\mathcal{C} = \{(1, 0, 0, 0, \omega, \omega^2), (0, 1, \omega, 0, 0, \omega), (0, 0, 1, \omega^2, 1, \omega^2),$$
$$(1, \omega, \omega, \omega^2, \omega^2, \omega^2), (1, \omega^2, \omega, \omega, 1, 0), (1, 1, 0, 1, 0, 0)\},$$

and the union $\mathcal{V} \cup \mathcal{C}$ is a non-complete 27-cap of PG(5, 4) whose stabilizer is the subgroup A_6 of the group PGU(3, 4) of $\mathcal{V}$.

We checked with MAGMA that no hyperplane of PG(5, 4) is disjoint from $\mathcal{V} \cup \mathcal{C}$. Hence, $\mathcal{V} \cup \mathcal{C}$ gives rise to a cap code which is a linear $[27, 6, 16]_4$-code admitting $C_3 \times A_6$ as its automorphism group, and the weight distribution of this code is

$$(0, 1), (16, 513), (18, 180), (20, 2160), (22, 864), (24, 270), (26, 108).$$

Table 1
The seven points of the Baer subplane π in PG(2, 4).

E_1(1,0,0),	E_2(0,1,0),	E_3(0,0,1),
B_1(1,1,1),	B_2(1,0,1),	B_3(0,1,1),
B_4(1,1,0).		

Table 2
The seven points of the Gale transform π' of π in PG(3, 4).

U_1(1,1,0,1),	U_2(1,0,1,1),	U_3(1,1,1,0),
E_1(1,0,0,0),	E_2(0,1,0,0),	E_3(0,0,1,0),
E_4(0,0,0,1).		

Remark 2.1. The Gale transform of a classical unital of PG(2, q) gives rise to a $(q\sqrt{q}+1)$-cap of PG$(q\sqrt{q}-3, q)$. Some computer tests suggest that for large q the resulting $(q\sqrt{q}+1)$-cap is never complete and its size is very small compared to the dimension of the target space. This makes the associated linear codes less interesting. On the other hand, it is not clear how to claim that the cap is embedded in a Veronese variety. Similar arguments apply when the unital is non-classical.

3. Further codes

As we observed in the previous section, the group $A_6 \cong$ PSL(2, 9) is the stabilizer of the set $\mathcal{V} \cup \mathcal{C}$. Take now all the orbits under the action of A_6 on PG(5, 4). Three of them, of lengths respectively 6, 6 and 15, yield a partition of $\mathcal{V} \cup \mathcal{C}$; in particular, $\mathcal{V}$ is partitioned into six points of a frame and its complement in $\mathcal{V}$. Other five orbits of lengths respectively 36, 36, 36, 45 and 45 are caps in PG(5, 4). The two long orbits produce isomorphic $[45, 6, 26]_4$-codes with weight distribution

$$(0, 1), (26, 135), (30, 441), (32, 270), (33, 1170), (34, 675),$$
$$(35, 648), (36, 405), (39, 180), (40, 108), (42, 45), (45, 18),$$

while the three short orbits produce two isomorphic $[36, 6, 20]_4$-codes with weight distribution

$$(0, 1), (20, 108), (24, 720), (26, 1296), (28, 1080), (30, 720), (32, 135), (36, 36),$$

and a $[36, 6, 22]_4$-code with weight distribution

$$(0, 1), (22, 270), (24, 855), (26, 324), (28, 2025), (30, 558), (36, 63).$$

Again, all of these codes admit $C_3 \times A_6$ as their automorphism group.

4. The Gale transform of a distinguished 10-set in PG(3, 4)

In PG(3, 4) with projective coordinates X_0, X_1, X_2, X_3 consider the following 5-set of points:

$$\mathcal{C} = \{(1, 1, 1, 0), (1, 0, 1, 1), (0, 1, 1, 1), (0, 0, 0, 1), (1, 1, \omega^2, \omega)\},$$

where ω is a primitive element of GF(4) such that $\omega^2 + \omega + 1 = 0$. It is easy to check that no four points of $\mathcal{C}$ are coplanar and hence $\mathcal{C}$ is a 5-arc in PG(3, 4). From [5, p. 248] $\mathcal{C}$ is actually a twisted cubic. The stabilizer G_4 of $\mathcal{C}$ in PGL(4, 4) is isomorphic to Sym(5) [5, Lemma 21.1.3(ii)]. The stabilizer of $P = (1, 1, 1, 0) \in \mathcal{C}$ in G_4, say H, is isomorphic to Sym(4). It fixes P and permutes the remaining points of $\mathcal{C}$ in a single orbit. Among the other orbits of H on points of PG(3, 4) there is a distinguished one, say O, of length 6 consisting of the points (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0), $(1, 1, \omega, 0)$, $(1, \omega, 1, 0)$ and $(1, \omega^2, \omega^2, 0)$ lying on the plane $X_3 = 0$, that is, the osculating plane to $\mathcal{C}$ at the point P. Gluing together the orbit O and the points of $\mathcal{C} \setminus \{P\}$ we obtain a 10-set $\mathcal{X}$ containing the fundamental frame. It turns out that the Gale transform of $\mathcal{X}$ lies on the Veronese surface $\mathcal{V}$ described in Section 2.

5. The code from a Baer subplane

In this Section we study the Gale transform of a Baer subplane $\mathcal{B}$ of PG(2, q), with q square. Certainly, it gives rise to a $(q + \sqrt{q} + 1)$-cap which is embedded in a Baer subgeometry PG$(q + \sqrt{q} - 3, \sqrt{q})$ of PG$(q + \sqrt{q} - 3, q)$. As the group PGL(3, q) acts transitively on the set of all Baer subplanes of PG(2, q), we can always assume, without loss of generality, that $\mathcal{B}$ is the canonical Baer subplane of PG(2, q), that is, its points have coordinates in GF$(\sqrt{q})$. As such, it is straightforward to see that the Gale transform of a Baer subplane determines a $(q + \sqrt{q} + 1)$-cap embedded in a suitable Baer subgeometry. We begin with the smallest case.

Let π = PG(2, 2) be the Baer subplane of PG(2, 4) containing the projective frame, that is, the point set of π is as in Table 1, while its Gale transform π' is a subset of PG(3, 4) lying in a Baer subgeometry as in Table 2.

The automorphism group of π and π' is G = PGL(3, 2). With the aid of MAGMA [2] we checked that G fixes a point and a plane in PG(3, 4). Further, the action of G on PG(3, 4) produces, among the others, three orbits of size 7. One of them

Table 3
The orbit $\mathcal{O}_1$ in PG(3, 4).

$P_{11}(1, \omega, \omega^2, \omega^2)$,	$P_{12}(0, 1, 1, \omega)$,	$P_{13}(1, \omega^2, \omega^2, \omega)$,
$P_{14}(1, \omega, \omega, \omega)$,	$P_{15}(0, 1, \omega, 1)$,	$P_{16}(0, 1, \omega^2, \omega^2)$,
$P_{17}(1, \omega^2, \omega, \omega^2)$.		

Table 4
The orbit $\mathcal{O}_2$ in PG(3, 4).

$P_{21}(0, 1, 1, \omega^2)$,	$P_{22}(1, \omega^2, \omega, \omega)$,	$P_{23}(1, \omega, \omega^2, \omega)$,
$P_{24}(0, 1, \omega^2, 1)$,	$P_{25}(1, \omega^2, \omega^2, \omega^2)$,	$P_{26}(1, \omega, \omega, \omega^2)$,
$_{27}(0, 1, \omega, \omega)$.		

Table 5
The nine points of $\mathcal{H}(2, 4)$ in PG(2, 4).

E_1(1,0,0),	E_2(0,1,0),	E_3(0,0,1),
$P_1(1, \omega, 0)$,	$P_2(0, 1\omega)$,	$P_3(1, \omega 1)$,
$P_4(1, \omega^2, \omega)$,	$P_5(1, \omega^2, 1)$,	$P_6(1, 0, \omega)$.

Table 6
The nine points of $\mathcal{H}'$ in PG(5, 4) after normalization to the left.

$Q_1(1, 0, 1, 1, 1, 1)$,	$Q_2(1, \omega^2, 1, \omega, \omega, 0)$,	$Q_3(0, 1, \omega^2, 1, \omega^2, 1)$,
$F_1(1, 0, 0, 0, 0, 0)$,	$F_2(0, 1, 0, 0, 0, 0)$,	$F_3(0, 0, 1, 0, 0, 0)$,
$F_4(0, 0, 0, 1, 0, 0)$,	$F_5(0, 0, 0, 0, 1, 0)$,	$F_6(0, 0, 0, 0, 0, 1)$.

Table 7
The twelve points of the orbit $\mathcal{R}$.

$V_1(1, 0, 0, 0, 1, 1)$,	$V_2(0, 1, 1, 0, 0, \omega^2)$,	$V_3(1, \omega^2, 1, 1, \omega, 1)$,
$V_4(1, \omega^2, 1, 1, \omega, 1)$,	$V_5(1, \omega^2, \omega^2, \omega, \omega^2, 0)$,	$V_6(0, 1, \omega^2, \omega, \omega, 1)$,
$V_7(0, 1, \omega^2, 0, 0, 1)$,	$V_8(1, 0, \omega, \omega^2, 1, 1)$,	$V_9(1, \omega^2, 0, \omega, 0, 0)$,
$V_{10}(1, \omega, 0, \omega^2, 0, 0)$,	$V_{11}(1, 1, 1, 1, \omega^2, 1)$,	$V_{12}(1, 0, 0, 0, \omega^2, \omega)$.

Table 8
The nine points of $\mathcal{S}$.

$\Pi_1(1, 1, 1, \omega, 1, \omega^2)$,	$\Pi_2(0, 1, 0, 1, \omega, \omega^2)$,	$\Pi_3(1, 1, \omega^2, \omega^2, \omega^2\omega)$,
$\Pi_4(0, 0, 1, \omega^2, 1, \omega^2)$,	$\Pi_5(1, \omega, \omega, \omega^2, \omega^2, \omega^2)$,	$\Pi_6(1, \omega, \omega^2, 0, \omega, 0)$,
$\Pi_7(1, 0, \omega, 1, 0, \omega)$,	$\Pi_8(1, 0, 1, \omega^2, \omega^2, 0)$,	$\Pi_9(0, 1, 1, \omega, \omega^2, 0)$.

Table 9
The twelve points of $\mathcal{T}$.

$R_1(1, \omega^2, \omega, \omega, 1, 0)$,	$R_2(1, \omega^2, 0, \omega^2, \omega^2, 1)$,	$R_3(1, \omega^2, \omega^2, 1, 0, \omega^2)$,
$R_4(1, \omega^2, 1, 0, \omega, \omega)$,	$R_5(1, \omega, 0, \omega, 1, \omega)$,	$R_6(1, 1, \omega, 0, \omega, 1)$.
$R_7(1, 0, 0, 0, \omega, \omega^2)$	$R_8(1, 0, \omega^2, \omega, 1, 1)$	$R_9(1, \omega, 1, 1, 0, 1)$
$R_{10}(0, 1, \omega^2, \omega^2, 1, 1)$	$R_{11}(1, 1, 0, 1, 0, 0)$	$R_{12}(0, 1, \omega, 0, 0, \omega)$

coincides with π' while the other two, say $\mathcal{O}_1$ and $\mathcal{O}_2$, are as in Tables 3 and 4 respectively. Every pairing of these three orbits produces a complete 14-cap in PG(3, 4) whose automorphism group is isomorphic to $\mathrm{PGL}(3, 2) \times A$, with $|A| = 8$.

The three cap codes obtained from the above pairings are isomorphic. More precisely, each of them is an even $[14, 4, 8]_4$-code with weight distribution

$$(0, 1), (8, 21), (10, 168), (12, 42), (14, 24)$$

admitting an automorphism group of order 4032 isomorphic to

$$\mathrm{PGL}(3, 2) \times A \times C_3.$$

Assume $q = 3$. Let $\pi = \mathrm{PG}(2, 3)$ be the Baer subplane of PG(2, 9) in canonical position. This time the Gale transform π' is a subset of PG(9, 3) and it represents the Veronese embedding of π with respect to the complete linear system of cubics of π. It turns out that π' is not complete as a cap. Something similar happens when $q = 5$. This time we get a 31-cap of PG(27, 5) representing the Veronese embedding of $\pi = \mathrm{PG}(2, 5)$ with respect to the complete linear system of curves of degree six. For higher values of q the picture seems to be much more wild. However, it is conceivable that for some particular higher values of the square order q the Gale transform of a Baer subplane π of PG(2, q) corresponds to the embedding of π with respect to the complete linear system of curves of a given degree.

Table 10
The twenty-one conics on $\mathcal{V}$.

$\mathcal{C}_1$:	$P_{1,1}(0, 0, 0, 0, 0, 1)$,	$P_{1,2}(1, \omega^2, 0, \omega, 0, 0)$,
$P_{1,3}(0, 1, \omega^2, \omega, \omega, 1)$,	$P_{1,4}(1, 0, \omega, \omega^2, 1, 1)$,	$P_{1,5}(1, 1, 1, 1, \omega^2, 1)$;
$\mathcal{C}_2$:	$P_{2,1}(1, \omega, 0, \omega^2, 0, 0)$,	$P_{2,2}(1, \omega^2, 1, \omega^2, \omega, \omega^2)$,
$P_{2,3}(1, 0, \omega, \omega^2, 1, 1)$,	$P_{2,4}(0, 1, 1, 0, 0, \omega^2)$,	$P_{2,5}(0, 0, 0, 0, 1, 0)$;
$\mathcal{C}_3$:	$P_{3,1}(1, 0, 0, 0, \omega^2, \omega)$,	$P_{3,2}(1, \omega^2, 0, \omega, 0, 0)$,
$P_{3,3}(0, 1, \omega^2, 0, 0, 1)$,	$P_{3,4}(1, \omega^2, 1, \omega^2, \omega, \omega^2)$,	$P_{3,5}(1, 0, 1, 1, 1, 1)$;
$\mathcal{C}_4$:	$P_{4,1}(0, 1, \omega^2, 0, 0, 1)$,	$P_{4,2}(0, 1, \omega^2, 1, \omega^2, 1)$,
$P_{4,3}(0, 0, 0, 1, 0, 0)$,	$P_{4,4}(0, 1, \omega^2, \omega, \omega, 1)$,	$P_{4,5}(0, 0, 0, 0, 1, 0)$;
$\mathcal{C}_5$:	$P_{5,1}(0, 0, 1, 0, 0, 0)$,	$P_{5,2}(1, 0, 0, 0, \omega^2, \omega)$,
$P_{5,3}(1, \omega, 0, \omega^2, 0, 0)$,	$P_{5,4}(0, 1, \omega^2, \omega, \omega, 1)$,	$P_{5,5}(1, \omega^2, 1, 1, \omega, 1)$;
$\mathcal{C}_6$:	$P_{6,1}(0, 0, 0, 0, 0, 1)$,	$P_{6,2}(1, \omega, 0, \omega^2, 0, 0)$,
$P_{6,3}(0, 1, \omega^2, 1, \omega^2, 1)$,	$P_{6,4}(1, \omega^2, \omega^2, \omega, \omega^2, 0)$,	$P_{6,5}(1, 0, 1, 1, 1, 1)$;
$\mathcal{C}_7$:	$P_{7,1}(1, 0, 0, 0, \omega^2, \omega)$,	$P_{7,2}(0, 0, 0, 1, 0, 0)$,
$P_{7,3}(1, \omega^2, \omega^2, \omega, \omega^2, 0)$,	$P_{7,4}(0, 1, 1, 0, 0, \omega^2)$,	$P_{7,5}(1, 1, 1, 1, \omega^2, 1)$;
$\mathcal{C}_8$:	$P_{8,1}(1, \omega^2, 0, \omega, 0, 0)$,	$P_{8,2}(0, 1, \omega^2, 1, \omega^2, 1)$,
$P_{8,3}(0, 1, 1, 0, 0, \omega^2)$,	$P_{8,4}(1, \omega^2, 1, 1, \omega, 1)$,	$P_{8,5}(1, 0, 0, 0, 1, 1)$;
$\mathcal{C}_9$:	$P_{9,1}(0, 0, 1, 0, 0, 0)$,	$P_{9,2}(0, 0, 0, 1, 0, 0)$,
$P_{9,3}(1, 0, \omega, \omega^2, 1, 1)$,	$P_{9,4}(1, 0, 1, 1, 1, 1)$,	$P_{9,5}(1, 0, 0, 0, 1, 1)$;
$\mathcal{C}_{10}$:	$P_{10,1}(0, 0, 1, 0, 0, 0)$,	$P_{10,2}(1, \omega^2, 0, \omega, 0, 0)$,
$P_{10,3}(1, \omega^2, \omega^2, \omega, \omega^2, 0)$,	$P_{10,4}(1, \omega^2, 1, \omega, \omega, 0)$,	$P_{10,5}(0, 0, 0, 0, 1, 0)$;
$\mathcal{C}_{11}$:	$P_{11,1}(1, 0, 0, 0, \omega^2, \omega)$,	$P_{11,2}(0, 1, \omega^2, 1, \omega^2, 1, \omega^2, 1)$,
$P_{11,3}(0, 1, 0, 0, 0, 0)$,	$P_{11,4}(1, 0, \omega, \omega^2, 1, 1)$,	$P_{11,5}(1, \omega^2, 1, \omega, \omega, 0)$;
$\mathcal{C}_{12}$:	$P_{12,1}(1, \omega, 0, \omega^2, 0, 0)$,	$P_{12,2}(0, 1, \omega^2, 0, 0, 1)$,
$P_{12,3}(1, 1, 1, 1, \omega^2, 1)$,	$P_{12,4}(1, \omega^2, 1, \omega, \omega, 0)$,	$P_{12,5}(1, 0, 0, 0, 1, 1)$;
$\mathcal{C}_{13}$:	$P_{13,1}(0, 0, 0, 0, 0, 1)$,	$P_{13,2}(0, 0, 0, 1, 0, 0)$,
$P_{13,3}(1, \omega^2, 1, \omega^2, \omega, \omega^2)$,	$P_{13,4}(1, \omega^2, 1, 1, \omega, 1)$,	$P_{13,5}(1, \omega^2, 1, \omega, \omega, 0)$;
$\mathcal{C}_{14}$:	$P_{14,1}(1, \omega^2, \omega^2, \omega, \omega^2, 0)$,	$P_{14,2}(0, 1, 0, 0, 0, 0)$,
$P_{14,3}(0, 1, \omega^2, \omega, \omega, 1)$,	$P_{14,4}(1, \omega^2, 1, \omega^2, \omega, \omega^2)$,	$P_{14,5}(1, 0, 0, 0, 1, 1)$;
$\mathcal{C}_{15}$:	$P_{15,1}(0, 1, 0, 0, 0, 0)$,	$P_{15,2}(1, 0, 1, 1, 1, 1)$,
$P_{15,3}(1, 1, 1, 1, \omega^2, 1)$,	$P_{15,4}(1, \omega^2, 1, 1, \omega, 1)$,	$P_{15,5}(0, 0, 0, 0, 1, 0)$;
$\mathcal{C}_{16}$:	$P_{16,1}(0, 1, \omega^2, 0, 0, 1)$,	$P_{16,2}(1, \omega^2, \omega^2, \omega, \omega^2, 0)$,
$P_{16,3}(1, 0, \omega, \omega^2, 1, 1)$,	$P_{16,4}(1, 0, 0, 0, 0, 0)$,	$P_{16,5}(1, \omega^2, 1, 1, \omega, 1)$;
$\mathcal{C}_{17}$:	$P_{17,1}(0, 0, 0, 0, 0, 1)$,	$P_{17,2}(1, 0, 0, 0, \omega^2, \omega)$,
$P_{17,3}(1, 0, 0, 0, 0, 0)$,	$P_{17,4}(0, 0, 0, 0, 1, 0)$,	$P_{17,5}(1, 0, 0, 0, 1, 1)$;
$\mathcal{C}_{18}$:	$P_{18,1}(0, 1, \omega^2, \omega, \omega, 1)$,	$P_{18,2}(1, 0, 0, 0, 0, 0)$,
$P_{18,3}(1, 0, 1, 1, 1)$,	$P_{18,4}(0, 1, 1, 0, 0, \omega^2)$,	$P_{18,5}(1, \omega^2, 1, \omega, \omega, 0)$;
$\mathcal{C}_{19}$:	$P_{19,1}(0, 0, 0, 0, 0, 1)$,	$P_{19,2}(0, 0, 1, 0, 0, 0)$,
$P_{19,3}(0, 1, \omega^2, 0, 0, 1)$,	$P_{19,4}(0, 1, 0, 0, 0, 0)$,	$P_{19,5}(0, 1, 1, 0, 0, \omega^2)$;
$\mathcal{C}_{20}$:	$P_{20,1}(0, 0, 1, 0, 0, 0)$,	$P_{20,2}(0, 1, \omega^2, 1, \omega^2, 1)$,
$P_{20,3}(1, \omega^2, 1, \omega^2, \omega, \omega^2)$,	$P_{20,4}(1, 0, 0, 0, 0, 0)$,	$P_{20,5}(1, 1, 1, 1, \omega^2, 1)$;
$\mathcal{C}_{21}$;	$P_{21,1}(1, \omega, 0, \omega^2, 0, 0)$,	$P_{21,2}(1, \omega^2, 0, \omega, 0, 0)$,
$P_{21,3}(0, 0, 0, 1, 0, 0)$,	$P_{21,4}(0, 1, 0, 0, 0, 0)$,	$P_{21,5}(1, 0, 0, 0, 0, 0)$.

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